

## Duality relation for frustrated spin models

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We consider discrete spin models on arbitrary planar graphs and lattices with frustrated interactions. We first analyze the Ising model with frustrated plaquettes. We use an algebraic approach to derive the result that an Ising model with some of its plaquettes frustrated has a dual model which is an Ising model with an external field  $i\pi/2$  applied to the dual sites centered at frustrated plaquettes. In the case where all plaquettes are frustrated, this leads to the known result that the dual model has a uniform field  $i\pi/2$ , whose partition function can be evaluated in the thermodynamic limit for regular lattices. The analysis is extended to a Potts spin glass with analogous results obtained.

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### I. THE FRUSTRATED ISING MODEL

A central problem in the study of lattice-statistical problems is the consideration of frustrated spin systems (see, for example, Refs. [1–4]). A particularly useful tool in the study of spin systems is the consideration of duality relations (see, for example, Refs. [5,6]). Here, we apply the duality consideration to frustrated discrete spin systems.

We consider first the Ising model on an *arbitrary* planar graph  $G$ . A planar graph is a collection of vertices and (non-crossing) edges. Place Ising spins at vertices of  $G$ , which interact with competing interactions along the edges. Denote the interaction between sites  $i$  and  $j$  by  $-J_{ij} = -S_{ij}J$ , where  $S_{ij} = \pm 1$  and  $J > 0$ . Then the Hamiltonian is

$$\mathcal{H}(\sigma; S) = - \sum_{\langle i,j \rangle} S_{ij} J \sigma_i \sigma_j, \quad (1)$$

where  $\sigma_i = \pm 1$  is the spin at the site  $i$  and the summation is taken over all interacting pairs.

The Hamiltonian (1) plays an important role in condensed matter physics and related topics. Regarding  $S_{ij}$  as a quenched random variable governed by a probability distribution, the Hamiltonian (1) leads to the Edwards-Anderson model of spin glasses [7]. By taking a different  $S_{ij}$ , the Hamiltonian becomes the Hopfield model of neural networks [8]. Here, we consider the Hamiltonian (1) with fixed plaquette frustrations.

Let  $G$  have  $N$  sites and  $E$  edges. Then it has

$$N^* = E + 2 - N \quad (\text{Euler relation}) \quad (2)$$

faces, including one infinite face containing the infinite region and  $N^* - 1$  internal faces which we refer to as plaquettes. The parity of a face is the product of the edge  $S_{ij}$  factors around the face which can be either  $+1$  or  $-1$ . A face is *frustrated* if its parity is  $-1$ . An Ising model is frustrated if any of its plaquettes is frustrated, and is fully (totally) frustrated if every plaquette is frustrated. The fully frustrated model is also known as the odd model of the spin

glass [1]. As the parity of the infinite face is the product of the parities of all plaquettes, the parity of the infinite face in a totally frustrated Ising model is  $-1$  for  $N^* = \text{even}$  and  $+1$  for  $N^* = \text{odd}$ . An example of a full frustration is the triangular model with  $S_{ij} = -1$  for all nearest neighbor sites  $i, j$ .

The values of parity associated with all plaquettes define a “parity configuration” which we denote by  $\Gamma$ . The set of interactions  $\{S_{ij}\}$  corresponding to a given  $\Gamma$  is not unique. For the triangular model, for example, any  $\{S_{ij}\}$  which has either one or three  $S_{ij} = -1$  edges around every plaquette is totally frustrated. For a given  $\{S_{ij}\}$  and  $\Gamma$ , the partition function is the summation

$$Z(\{S_{ij}\}) = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \prod_E e^{S_{ij} J \sigma_i \sigma_j}, \quad (3)$$

where the product is taken over the  $E$  edges of  $G$ .

#### A. Gauge transformation

A *gauge* transformation is site-dependent redefinition of the up (down) spin directions. Mathematically, a gauge transformation transforms the spin variables according to [2]

$$\sigma_i \rightarrow \sigma'_i = w_i \sigma_i, \quad i = 1, \dots, N. \quad (4)$$

In the above, if  $w_i = +1$ , the original definition of up (down) spin directions is maintained, and if  $w_i = -1$ , the definitions of up (down) are exchanged. Under the gauge transformation, the  $S_{ij}$  in Eq. (1) transforms as follows:

$$S_{ij} \rightarrow S'_{ij} = w_i S_{ij} w_j \quad \forall i, j. \quad (5)$$

Since  $w_i^2 = 1$ , we have

$$\mathcal{H}(\sigma; S) = \mathcal{H}(\sigma'; S'). \quad (6)$$

Clearly, the gauge transformation (5) leaves the parity configuration  $\Gamma$  unchanged, i.e.,

$$\prod_{\text{face}} S_{ij} = \prod_{\text{face}} S'_{ij} \quad \forall \text{ face.} \quad (7)$$

For each parity configuration  $\Gamma$ , there are  $2^{N-1}$  different  $\{S_{ij}\}$  patterns consistent with it. To see this we note in Eq. (5), each of the  $2^N$  choices of  $\{w_i\}$  leads to a new  $\{S'_{ij}\}$  except the negation of all  $w_i$  which leaves  $\{S_{ij}\}$  unchanged. Conversely, any two sets of interactions  $\{S_{ij}\}$  and  $\{S'_{ij}\}$  for the same  $\Gamma$  are related by a gauge transformation which can be constructed as follows. Starting from any spin, say spin 1, assign the value  $w_1 = +1$ . One next builds up the graph by adding one site (and one edge) at a time. To the site 2 connected to 1 by the edge  $\{12\}$ , one assigns the factor  $w_2 = w_1 S_{12} S'_{12}$ , which yields  $w_1 S_{12} w_2 = S'_{12}$  consistent to Eq. (5). Proceeding in this way around a plaquette until an edge, say  $\{n1\}$ , completes a plaquette. At this point, one has

$$w_n S_{n1} w_1 = \left( \prod_{\text{plaquette}} S_{ij} \right) \left( \prod_{\text{plaquette}} S'_{ij} \right) S'_{n1} = S'_{n1}, \quad (8)$$

which is again consistent to Eq. (5). Continuing in this way, one constructs all  $w_i$  which transform  $\{S_{ij}\}$  into  $\{S'_{ij}\}$ . Note that if we had started with  $w_1 = -1$ , we would have resulted in the negation of all  $w_i$ . Thus, the bijection between the  $2^{N-1}$  sets  $\{S_{ij}\}$  and  $2^{N-1}$  gauge transformations is one to one.

In addition to Eq. (7), the gauge transformation also leaves the partition function invariant [9,10], i.e.,

$$Z(\{S_{ij}\}) = \sum_{\{\sigma\}} \prod_E e^{S_{ij} J \sigma_i \sigma_j} = \sum_{\{\sigma'\}} \prod_E e^{S'_{ij} J \sigma'_i \sigma'_j} = Z(\{S'_{ij}\}). \quad (9)$$

As a result, the partition function only depends on  $\Gamma$  and we can rewrite Eq. (3) as

$$Z(\Gamma) = 2^{-(N-1)} \sum_{\{S_{ij}\}} Z(\{S_{ij}\}), \quad (10)$$

where the summation is over all  $2^{N-1}$  distinct  $\{S_{ij}\}$  consistent with the parity configuration  $\Gamma$  for the same partition function. This expression of the partition function is used to derive the duality relation in ensuing sections.

### B. The fully frustrated Ising model

For our purposes, it is instructive to consider first the case of full frustration. Duality properties of fully frustrated model have previously been considered by a number of authors [2,4] for regular lattices. We present here an alternate formulation applicable to arbitrary graphs and arbitrary frustration.

The graph  $D$  dual to  $G$  has  $N^*$  sites each residing in a face of  $G$ , and  $E$  edges each intersecting an edge of  $G$ . We restrict to  $N^* = \text{even}$  so that all faces of  $G$  including the infinite face are frustrated. This restriction has no effect on the taking of the thermodynamic limit in the case of regular lattices. Since the signs  $S_{ij}$  around each face are subject to the constraint  $\prod S_{ij} = -1$ , we introduce in the summand of Eq. (10) a face

factor  $(1 - \prod S_{ij})/2$  and sum over  $S_{ij} = \pm 1$  independently. Similarly, writing  $\sigma_{ij} = \sigma_i \sigma_j$ , we can replace summations over  $\sigma_i = \pm 1$  in Eq. (10) by  $\sigma_{ij} = \pm 1$  by introducing a factor  $(1 + \prod \sigma_{ij})/2$  to each face. Thus, Eq. (10) becomes

$$Z_{\text{FF}} = \left[ \frac{2 \cdot 2^{-(N-1)}}{2^{2N^*}} \right] \sum_{\{\sigma_{ij}\}} \sum_{\{S_{ij}\}} \left[ \prod_E e^{S_{ij} \sigma_{ij} J} \right] \times \prod_{\text{face}} \left[ \left( 1 + \prod \sigma_{ij} \right) \left( 1 - \prod S_{ij} \right) \right], \quad (11)$$

where the subscript FF denotes full frustration, and the extra factor 2 in Eq. (11) is due to the  $2 \rightarrow 1$  mapping from  $\sigma_i$  to  $\sigma_{ij}$ .

For a face having  $n$  sides, we rewrite the face factors as

$$1 + \prod \sigma_{ij} = \sum_{\mu = \pm} \prod F(\sigma_{ij}; \mu), \quad (12)$$

$$1 - \prod S_{ij} = \sum_{\nu = \pm} \prod G(S_{ij}; \nu), \quad (13)$$

where each product has  $n$  factors

$$F(\sigma; \mu) = \delta_{\mu+} + \sigma \delta_{\mu-}, \quad (14)$$

$$G(S; \nu) = \delta_{\nu+} + S \omega_n \delta_{\nu-},$$

$\delta$  is the Kronecker delta function, and  $\omega_n = (-1)^{-1/n} = e^{-i\pi/n}$ .

We now regard  $\mu$  and  $\nu$  as indices of two Ising spins residing at each dual lattice site. After carrying out summations over  $\sigma_{ij}$  and  $S_{ij}$ , the partition function (11) becomes

$$Z_{\text{FF}} = 2^{-E-N^*} \sum_{\{\mu\}} \sum_{\{\nu\}} \prod_E B(\mu, \nu; \mu', \nu'), \quad (15)$$

where we have made use of the Euler relation (2) and  $B$  is a Boltzmann factor given by

$$B(\mu, \nu; \mu', \nu') = \sum_{\sigma = \pm 1} \sum_{S = \pm 1} e^{S \sigma J} F(\sigma; \mu) \times F(\sigma; \mu') G(S; \nu) G(S; \nu'). \quad (16)$$

Here,  $G(S; \nu')$  is given by Eq. (14) with  $\omega_n \rightarrow \omega_{n'}$ ,  $= e^{-i\pi/n'}$  and the two faces containing spins  $\{\mu, \nu\}$  and  $\{\mu', \nu'\}$  have, respectively,  $n$  and  $n'$  sides.

Substituting Eq. (14) into Eq. (16) and making use of the identities

$$\begin{aligned} \delta_{\mu+} \delta_{\mu'+} + \delta_{\mu-} \delta_{\mu'-} &= (1 + \mu \mu')/2, \\ \delta_{\mu+} \delta_{\mu'-} + \delta_{\mu-} \delta_{\mu'+} &= (1 - \mu \mu')/2, \end{aligned} \quad (17)$$

one obtains

$$B(\mu, \nu; \mu', \nu') = 2A(1 + \mu \mu') \cosh J + 2B(1 - \mu \mu') \sinh J, \quad (18)$$

where

$$\begin{aligned}
A &= \delta_{\nu+} \delta_{\nu'+} + \omega_n \omega_{n'} \delta_{\nu-} \delta_{\nu'-}, \\
B &= \omega_n \delta_{\nu-} \delta_{\nu'+} + \omega_{n'} \delta_{\nu+} \delta_{\nu'-}.
\end{aligned} \tag{19}$$

We number the four states  $\{\mu, \nu\} = \{+, +\}, \{-, -\}, \{-, +\}, \{+, -\}$  by 1,2,3,4, respectively. The Boltzmann factor (18) can be conveniently written as a  $4 \times 4$  matrix

$$B(\mu, \nu; \mu', \nu') = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{pmatrix}, \tag{20}$$

where

$$\begin{aligned}
B_{11} &= 4 \cosh J, & B_{12} &= 4 \omega_{n'} \sinh J, \\
B_{21} &= 4 \omega_n \sinh J, & B_{22} &= 4 \omega_n \omega_{n'} \cosh J.
\end{aligned} \tag{21}$$

Thus, the partition function of the  $\{\mu, \nu\}$  spin model is twice that of an Ising model on the dual lattice. The exchange coupling constant  $K$  and the magnetic field  $h$  in the dual model are determined by

$$\begin{aligned}
B_{11} &= D e^{K+(h/n)+(h'/n')}, & B_{12} &= D e^{-K+(h/n)-(h'/n')}, \\
B_{21} &= D e^{-K-(h/n)+(h'/n')}, & B_{22} &= D e^{K-(h/n)-(h'/n')}.
\end{aligned} \tag{22}$$

Here,  $n$  and  $n'$  are the number of edges incident at the two dual sites, respectively.

The solution of the above equations gives

$$\begin{aligned}
e^{-2K} &= \tanh J > 0, & D &= 4(\omega_n \omega_{n'})^{1/2} \sqrt{\sinh J \cosh J}, \\
e^{2(h/n)} &= 1/\omega_n = e^{i\pi/n}, & e^{2(h'/n')} &= 1/\omega_{n'} = e^{i\pi/n'},
\end{aligned} \tag{23}$$

or equivalently

$$K = -\frac{1}{2} \ln(\tanh J) \quad \text{and} \quad h = h' = \frac{i\pi}{2}. \tag{24}$$

Thus, we have established the equivalence

$$Z_{\text{FF}}(J) = 2^{N-1} i^{N^*} (\sinh J \cosh J)^{E/2} Z_{\text{Ising}}^{(D)}\left(i \frac{\pi}{2}, K\right), \tag{25}$$

where  $Z_{\text{Ising}}^{(D)}(i\pi/2, K)$  is the partition function of a ferromagnetic Ising model on  $D$  with interactions  $K > 0$  and an external field  $i\pi/2$ . In writing down Eq. (25), we have made use of the identity  $2 \times 2^{-(E+N^*)} 4^E = 2^{N-1}$  and the fact that  $(\omega_n \omega_{n'})^{E/2} = (-i)^{N^*} = i^{N^*}$  for  $N^* = \text{even}$ . We make the following remarks:

(1) The duality relation (25) has previously been obtained by Fradkin *et al.* [2], and for the square lattice by Suzuki [4] and Sütő [11], and by Au-Yang and Perk [12] in another context.

(2) The duality relation (25) is different from the Kadanoff-Ceva-Merlini scheme [13,14] of replacing  $K$  by  $K + i\pi/2$  [corresponding to  $J < 0$  in Eq. (23)] in the ferro-

magnetic model. Suzuki [4] has made the explicit use of the Kadanoff-Ceva-Merlini scheme in deriving Eq. (25) for the square lattice. For fully frustrated systems, the Suzuki method can be extended to any graph whose dual admits dimer coverings.

(3) The duality relation (25) holds for a fixed  $\{S_{ij}\}$  without probability considerations and, therefore, differs intrinsically from that of a spin glass obtained recently by Nishimori and Nemoto [15] using a replica formulation.

(4) The duality relation (25) which holds for any lattice appears to support the suggestion [3] that all fully frustrated Ising models belong to the same universality class.

### C. The thermodynamic limit

The partition function (25) for an Ising model in a uniform field  $i\pi/2$  can be exactly evaluated for regular lattices. Defining the per-site ‘‘free energy’’

$$f = \lim_{N^* \rightarrow \infty} \frac{1}{N^*} \ln Z_{\text{Ising}}^{(D)}\left(i \frac{\pi}{2}, K\right), \tag{26}$$

Lee and Yang [16] have obtained a closed form expression of  $f(K)$  for the square lattice. Their result, which was later derived rigorously by McCoy and Wu [17] and others [14,18], is

$$\begin{aligned}
f &= i \frac{\pi}{2} + C + \frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ln[z + z^{-1} + 2 \\
&\quad - 4 \cos \theta \cos \phi],
\end{aligned} \tag{27}$$

where  $C = [\ln(\sinh 2K)]/2$ ,  $z = e^{-4K}$ . The free energy (27), which is the same as that obtained by Villain [1], is analytic at all nonzero temperatures.

The solution for the triangular Ising model in a field  $i\pi/2$  has also been deduced previously [18,19]. However, it can also be obtained most simply by observing that the honeycomb lattice, which is the dual of the triangular lattice, has a coordination number 3. It follows that we can recast the field Boltzmann weights as  $e^{i\pi\sigma_j/2} = i\sigma_j = i\sigma_j^3$  and redistribute the  $\sigma_j^3$  factor at site  $j$  to its three incident edges. Then, as pointed out by Suzuki [4], each edge can be associated with a factor  $i\sigma_i\sigma_j e^{K\sigma_i\sigma_j} = e^{(K+i\pi/2)\sigma_i\sigma_j}$  and the desired solution can be obtained from that of the *zero-field* honeycomb lattice with the simple replacement  $K \rightarrow K + i\pi/2$ . This gives

$$\begin{aligned}
f &= i \frac{\pi}{2} + C + \frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ln[(1 + e^{4K})^2 \\
&\quad + 4 \cos \phi (\cos \theta + \cos \phi)],
\end{aligned} \tag{28}$$

where  $C = [\ln(2 \sinh 2K)]/2$  and  $K$  is the Ising interaction on the honeycomb lattice. Again, there is no finite temperature phase transition.

### D. Arbitrary plaquette parities

In a similar fashion, one can extend the above analysis to Ising models with arbitrary face parities. All the steps of previous sections can be carried out, except that for faces that are not frustrated, we must replace  $\omega_n$  by 1 at the corresponding dual sites. This results in a zero field (instead of a field  $i\pi/2$ ) at these sites. Thus, for an Ising model with arbitrary parity configuration  $\Gamma$ , its dual model has fields 0 and  $i\pi/2$ , respectively, at sites in faces of parity  $+1$  and  $-1$ . Explicitly, we find

$$Z(\Gamma) = 2^{N-1} (-i)^{N_F} (\sinh J \cosh J)^{E/2} Z_{\text{Ising}}^{(D)}(\{h_j\}, K), \quad (29)$$

where the dual partition function is

$$Z_{\text{Ising}}^{(D)}(\{h_j\}, K) = \sum_{\{\mu_i\}} \prod_E e^{K\mu_i\mu_j} \prod_{\text{face}} e^{h_i\mu_i}. \quad (30)$$

Here,  $N_F$  is the number of frustrated faces and the external field at site  $j$  is  $h_j = i\pi/2$  or 0 depending on whether the face associated with the site is frustrated or not. We give the following remarks:

(1) The duality relation (30) for Ising models with arbitrary frustrated plaquettes can be found as contained implicitly in Ref. [2].

(2) By writing  $e^{i\pi\sigma/2} = i\sigma$  in the dual partition function, we see that the partition function of an Ising model with  $p$  frustrated faces is dual to a  $p$ -spin Ising correlation function in zero field. In particular, the  $p=2$  correlation problem has been studied in detail [17], which now leads to a wealth of information on the correlation of two frustrated plaquettes.

## II. POTTS SPIN GLASS

Our analysis can be extended to a  $q$ -state spin model, the Potts spin glass. First, we recall the definition of a chiral Potts model. The chiral Potts model, which was considered in Ref. [5], is a discrete  $q$ -state spin model with a cyclic Boltzmann factor  $\Lambda(\xi, \xi') = \Lambda(\xi - \xi')$  between two spins at sites  $i$  and  $j$  and in states  $\xi_i$  and  $\xi_j = 0, 1, \dots, q-1$ . The interactions are  $q$  periodic, namely, the Boltzmann factor satisfies

$$U(\xi + q) = U(\xi). \quad (31)$$

The interaction can be symmetric, namely,  $U(\xi) = U(-\xi)$ , as in the case of the standard Potts model [20], but in our considerations, this need not be the case.

A Potts spin glass is a chiral Potts model with random interactions. To describe the randomness, one introduces edge variables  $\lambda_{ij} = \lambda_{ji} = 0, 1, \dots, q-1$  and considers the partition function [15,21,22]

$$Z_{\text{Potts}}(\{\lambda_{ij}\}) = \sum_{\xi_i=0}^{q-1} \prod_E U(\xi_i - \xi_j + \lambda_{ij}). \quad (32)$$

Note that if  $U$  is symmetric and  $q=2$ , the partition function (32) reduces to Eq. (3). A plaquette has ‘‘flux’’  $r$  ( $= 0, 1, 2, \dots, q-1$ ) if [23]

$$\sum_{\text{plaq}} \lambda_{ij} = r \pmod{q}. \quad (33)$$

A set of  $\{\lambda_{ij}\}$  leads to a flux configuration  $\Gamma$ , which is specified by the values of the flux for all faces.

### A. Gauge transformation

A gauge transformation for the Potts spin glass is the mapping

$$\begin{aligned} \xi_i &\rightarrow \xi'_i = \xi_i + \theta_i, \\ \lambda_{ij} &\rightarrow \lambda'_{ij} = \lambda_{ij} + \theta_i - \theta_j, \end{aligned} \quad (34)$$

where  $\theta_i = 0, 1, \dots, q-1$ . It is clear that this mapping leaves the flux configuration  $\Gamma$  unchanged, i.e.,

$$\sum_{\text{plaq}} \lambda_{ij} = \sum_{\text{plaq}} \lambda'_{ij}. \quad (35)$$

Since a global change of all  $\xi_i$  by the same amount preserves  $\{\lambda_{ij}\}$ , the total number of distinct  $\{\lambda_{ij}\}$  consistent with a particular flux configuration  $\Gamma$  is  $q^{N-1}$ . Conversely, any two sets of  $\{\lambda_{ij}\}$  and  $\{\lambda'_{ij}\}$  giving rise to the same flux configuration are related through a gauge transformation. To see this, we start from an arbitrarily chosen site, say site 1, and set  $\theta_1 = 0$ . Next, we assign  $\theta_2 = \theta_1 + \lambda_{12} - \lambda'_{12}$  to site 2 connected to site 1 by an edge. Continuing in this way as in the Ising case, one eventually determines a set of  $\theta_i$  that transforms  $\{\lambda_{ij}\}$  into  $\{\lambda'_{ij}\}$ , and vice versa. The bijection between the  $q^{N-1}$  configurations of  $\lambda_{ij}$  and gauge transformations for a given  $\Gamma$  is one to one.

In addition to leaving the flux configuration unchanged, gauge transformation also leaves the partition function invariant, namely,

$$Z_{\text{Potts}}(\{\lambda_{ij}\}) = Z_{\text{Potts}}(\{\lambda'_{ij}\}). \quad (36)$$

Thus, analogous to Eq. (10), we have

$$Z_{\text{Potts}}(\Gamma) = q^{-(N-1)} \sum_{\{\lambda_{ij}\}} Z(\{\lambda_{ij}\}), \quad (37)$$

which is used to derive a duality relation. Again, the sum in Eq. (37) runs through all  $\{\lambda_{ij}\}$  consistent with a given flux configuration  $\Gamma$ .

### B. Duality relation

In the Potts partition function (32), we write  $\xi_{ij} = \xi_i - \xi_j$ , and to each face having a flux  $r$ , we introduce two factors,

$$\begin{aligned} &\frac{1}{q} \sum_{\mu=0}^{q-1} e^{-i2\pi\mu(\xi_{12} + \xi_{23} + \dots + \xi_{n1})/q} \\ &\times \begin{cases} 1 & \text{if } \xi_{12} + \dots + \xi_{n1} = 0 \pmod{q} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\frac{1}{q} \sum_{\nu=0}^{q-1} e^{-i2\pi\nu(\lambda_{12}+\lambda_{23}+\dots+\lambda_{n_1-r})/q} \times \begin{cases} 1 & \text{if } \lambda_{12}+\dots+\lambda_{n_1}=r \pmod{q} \\ 0 & \text{otherwise.} \end{cases} \quad (38)$$

This permits us to sum over  $\xi_{ij}$  and  $\lambda_{ij}$  independently. Thus, analogous to Eq. (15), we obtain

$$Z_{\text{Potts}}(\Gamma) = q^{-E-N^*} \sum_{\{\mu\}} \sum_{\{\nu\}} B(\mu, \nu; \mu', \nu'), \quad (39)$$

where

$$B(\mu, \nu; \mu', \nu') = \sum_{\lambda} \sum_{\xi} U(\xi + \lambda) \exp \left[ \frac{i2\pi}{q} \left( -(\mu - \mu')\xi - (\nu - \nu')\lambda + \frac{r\nu}{n} + \frac{r'\nu'}{n'} \right) \right], \quad (40)$$

and  $n, n'$  are the numbers of sides of the two plaquettes containing  $\{\mu, \nu\}$  and  $\{\mu', \nu'\}$ , and fluxes  $r$  and  $r'$ , respectively.

We carry out the summations in Eq. (40) after introducing the Fourier transform

$$U(\xi + \lambda) = \frac{1}{q} \sum_{\eta=0}^{q-1} \Lambda(\eta) e^{i2\pi\eta(\xi + \lambda)/q}, \quad (41)$$

where  $\Lambda(\eta)$  are the eigenvalues of the matrix  $U$  [5]. One obtains

$$B(\mu, \nu; \mu', \nu') = q \delta_{\mu - \mu', \nu - \nu'} \Lambda(\nu - \nu') \times e^{i2\pi r \nu / q n} e^{i2\pi r' \nu' / q n'}. \quad (42)$$

In the above equation,  $\delta_{\mu - \mu', \nu - \nu'}$  sets  $\mu - \mu'$  to  $\nu - \nu' \pmod{q}$ .

The substitution of Eq. (42) into Eq. (39) followed by summing over  $\mu$  now yields the result

$$Z_{\text{Potts}}(\Gamma) = q^{1-N^*} Z_{\text{Potts}}^{(D)}(\{h_j\}, \Lambda), \quad (43)$$

where

$$Z_{\text{Potts}}^{(D)}(\{h_j\}, \Lambda) = \sum_{\{v_i\}} \left[ \prod_E \Lambda(v_i, v_j) \right] \left[ \prod_{\text{face}} e^{h_j v_j} \right] \quad (44)$$

is the partition function of a chiral Potts model on the dual graph  $D$ , which generalizes Eq. (29) to Potts spin glasses. The dual chiral Potts model has Boltzmann weights  $\Lambda(\mu_i, \mu_j) = \Lambda(\mu_i - \mu_j)$  and external fields

$$h_j = i \frac{2\pi r_j}{q}, \quad j = 1, 2, \dots, N^*, \quad r_j = 0, 1, \dots, q-1 \quad (45)$$

on the spin in plaquette  $j$  which has a flux  $r_j$ . When  $r_j = 0$  for all  $j$ , Eq. (43) reduces to the duality relation for the zero-field chiral Potts model given by Eq. (13) in Ref. [5].

### III. SUMMARY

We have obtained duality relations for planar Ising and chiral Potts models on arbitrary graphs and with fixed plaquette parity or flux configurations. Our main results are the equivalences (25) for the fully frustrated Ising model, Eq. (29) for the Ising model with arbitrary plaquette parity, and Eq. (43) for the chiral Potts model with arbitrary flux configurations. In all cases, the dual models have pure imaginary fields applied to spins in plaquettes that are frustrated and/or having a nonzero flux.

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